# ECE106 practice problems

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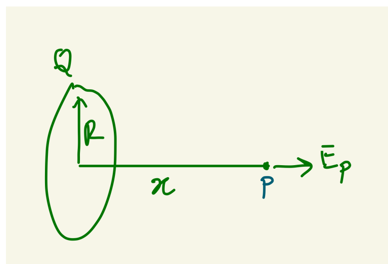
Disclaimer: Some problems are inherently hard analytically. The idea is to apply basic concepts in more creative problem settings, but not always end with complete closed-form solutions

1. (*Electric field outside ring charge*) Calculate the electric field outside a circular ring having uniform charge $Q$. Ring has radius $R$, and the point of interest (P) is on the axis of the ring at distance $x$ from the center of the ring.




*Solution.* This is a small recap of a primitive you should be familiar with. Convince yourself that

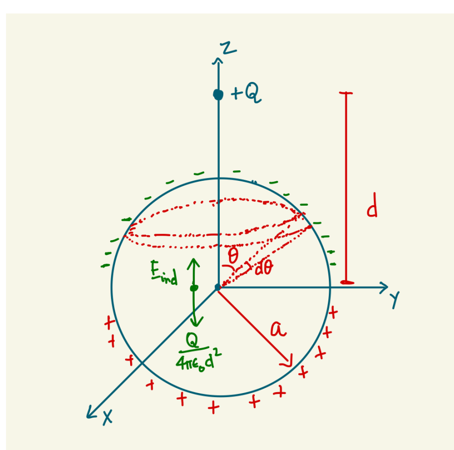
$$\vec{E}_P = \frac{Q}{4\pi\epsilon_0} \cdot \frac{x}{(x^2 + R^2)^{3/2}}\ \widehat{x}$$

2. (*Spherical conductor induced charge*) A point charge $+Q$ is placed outside a neutral, solid spherical conductor at distance $d$ from the sphere center. The sphere has radius $a < d$. If the point charge is placed on +Z axis, we can denote the induced surface charge density on the conductor as $\sigma(\theta)$, where $\theta$ is the standard spherical coordinate, range $[0, \pi]$.

i) The conductor is net-charge neutral. Charge neutrality gives a simple math constraint on $\sigma(\theta)$. Find out what it is.

ii) The net electric field is zero everywhere inside a conductor. Enforcing this at the center of the sphere gives an integral constraint on $\sigma(\theta)$. Find out what it is.

iii) Generalize part ii) to obtain similar integral constraints on $\sigma(\theta)$ at points $(0, 0, a)$ and $(0, 0, -a)$.



*Solution.* As the question suggests, we will not obtain a full closed-form solution for $\sigma(\theta)$. But we'd like to see how close we can get with constraints on its form. At an angle $\theta$ as shown in the figure, infinitesimal ring charge is given by

$$\begin{aligned} dq(\theta) &= \sigma(\theta)\cdot 2\pi a\sin\theta\cdot a\ d\theta \qquad (1)\\ &= 2\pi a^2\cdot\sigma(\theta)\sin\theta\ d\theta \qquad (2)\end{aligned}$$

i) Charge neutrality in math language translates to

$$\int_0^\pi dq(\theta) = 0$$
$$\Rightarrow \int_0^\pi \sigma(\theta)\sin\theta\ d\theta = 0$$

Since sine function is symmetric about $\theta = \pi/2$, this simply demands that $\sigma(\theta)$ be anti-symmetric about $\theta = \pi/2$. We finally have

$$\boxed{\sigma(\pi/2-\alpha) = \sigma(\pi/2+\alpha) \quad \text{for} \quad \alpha\in[0,\pi/2]} \qquad (3)$$

This in particular means $\sigma(\pi/2) = 0$, and $\sigma(0) = -\sigma(\pi) < 0$. As you see, charge neutrality already gives you much info about how a plot of $\sigma(\theta)$ vs $\theta$ should look like.

ii) How do we find the induced electric field at the centre of the sphere? Break the sphere into infinitesimal ring charges and use what we already know about a ring charge from Problem 1. And then force the integral you get to be equal to the field created by point charge $+Q$ at that point, in order to satisfy zero net Electric field in a conductor.

$$E_{ind} = \int_0^\pi dE_{ind}(\theta) = -\frac{Q}{4\pi\epsilon_0 d^2} \qquad (4)$$

Using the result from Problem 1 (ring charge), we see that

$$dE_{ind}(\theta) = \frac{dq(\theta)a\cos\theta}{4\pi\epsilon_0 a^3}$$

After simplification, the final constraint resulting from equation (4) is

$$\boxed{\text{At } z=0\text{:} \quad \int_0^{\pi/2}\sigma(\theta)\sin(2\theta)\ d\theta = -\frac{Q}{2\pi d^2}} \qquad (5)$$

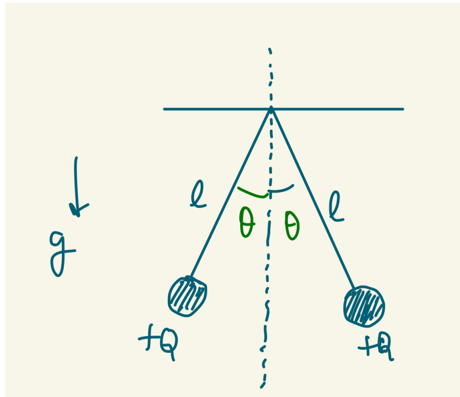
iii) The above strategy can be generalized to any point on the Z-axis to get additional constraints on $\sigma(\theta)$. In particular, it is interesting to attempt the calculation at points $(0,0,+a)$ and $(0,0,-a)$

Try the calculation yourself. It would be helpful to be aware of double-angle trig identities $\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$ and $\cos\theta = 2\cos^2(\theta/2) - 1 = 1 - 2\sin^2(\theta/2)$. In the end, I get the following

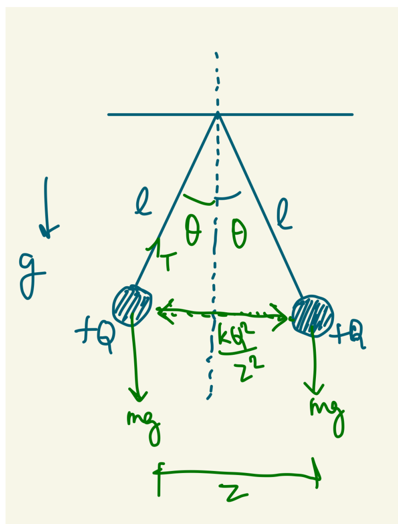
$$\boxed{\text{At } z=+a\text{:} \quad \int_0^{\pi}\sigma(\theta)\sin(\theta/2)\ d\theta = \frac{Q}{2\pi(d+a)^2}} \qquad (6)$$

$$\boxed{\text{At } z=-a\text{:} \quad \int_0^{\pi}\sigma(\theta)\cos(\theta/2)\ d\theta = -\frac{Q}{2\pi(d-a)^2}} \qquad (7)$$

3. (*Charges with mass in gravity*) Equal positive charges $+Q$ are suspended using length $l$ threads from a wall as shown in the figure. The charges have equal mass $m$ and are under the influence of earth's gravity denoted $g$. Find a constraint on the angle $\theta$ shown in the figure




*Solution.* Consider the following force balance picture




Tension in the thread is denoted $T$, distance between charges $z$. For either charge, we have two constraints from force balance

$$T \cos\theta = mg \tag{8}$$

$$T \sin\theta = \frac{Q^2}{4\pi\epsilon_0 z^2} \tag{9}$$

We eliminate tension by dividing the constraints, and see

$$z^2 \tan\theta = \frac{Q^2}{4\pi\epsilon_0 mg} \tag{10}$$

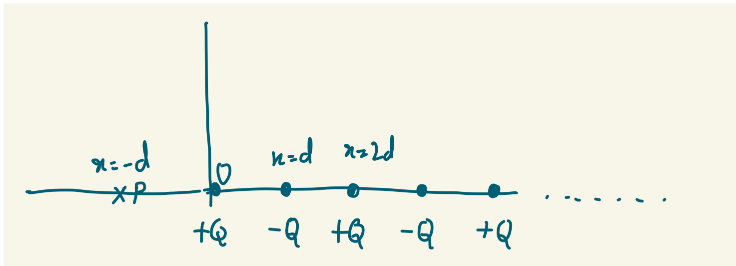
Observe that $z = 2l\sin\theta$. Hence we have

$$\boxed{\tan\theta \sin^2\theta = \frac{Q^2}{16\pi\epsilon_0 \cdot mg \cdot l^2}} \tag{11}$$

4. *(Infinite sequence of point charges)* Infinitely many point charges $+Q$ and $-Q$ are kept fixed in an alternating fashion on the +X axis as shown in the figure

i) What is the Electric Field at $x = -d$?

ii) What is the Electric Potential at $x = -d$?




Hints:

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} \dots \quad |x| \le 1$

$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

*Solution.* The idea is to use superposition principle along with interesting math results derived from convergent infinite series.

i) We have

$$\vec{E}(x=-d) = -\widehat{x}\ \frac{Q}{4\pi\epsilon_0 d^2}\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} \dots\right)$$

Let us the denote the infinite series above $S$. From the math hint, note that

$$\frac{\pi^2}{6} = \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \dots\right) + \frac{\pi^2}{24}$$
$$\Rightarrow \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \dots\right) = \frac{\pi^2}{8}$$

Therefore

$$S = \frac{\pi^2}{8} - \frac{\pi^2}{24} = \frac{\pi^2}{12}$$

Hence we get

$$\boxed{\vec{E}(x=-d) = -\widehat{x}\ \frac{Q}{4\pi\epsilon_0 d^2} \cdot \frac{\pi^2}{12} = -\frac{Q\pi}{48\epsilon_0 d^2}\ \widehat{x}} \tag{12}$$

ii) We have

$$V(x=-d) = \frac{Q}{4\pi\epsilon_0 d}\left(1 - \frac{1}{2} + \frac{1}{3} \dots\right)$$

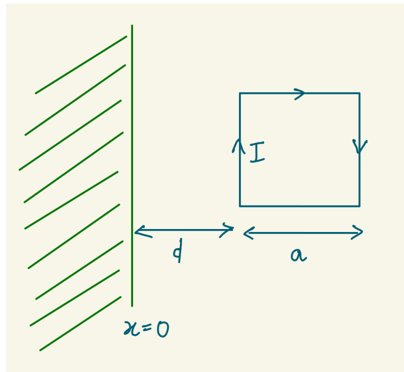
From the math hint, the infinite series above converges as

$$1 - \frac{1}{2} + \frac{1}{3} \dots = \ln 2$$

Hence we get

$$\boxed{V(x=-d) = \frac{Q}{4\pi\epsilon_0 d} \ln 2} \tag{13}$$

5. *(Magnetic flux trickery)* A current-carrying ($I$) square loop with side $a$ is placed at a distance $d$ from Y-axis ($x = 0$) in XY plane as shown in the figure. Calculate the net magnetic flux created by the loop on the entire left-half-plane $x < 0$.

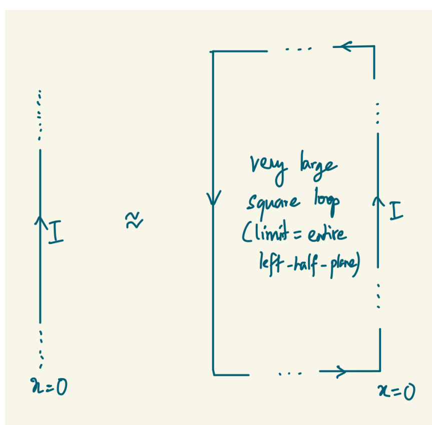



*Solution.* Call the left-half-plane ($x < 0$) Region-1 and the the square loop Region-2.

Hypothetically, imagine there was an infinite line wire placed on the Y-axis carrying the same current $I$. What would be the magnetic flux it creates on the square loop? It is not hard to see that

$$\phi_{1\to 2} = \int_{x=d}^{x=a+d} \frac{\mu_0 I}{2\pi x} \cdot a \; dx = \frac{\mu_0 I a}{2\pi} \ln\left(1 + \frac{a}{d}\right)$$

Now imagine the infinite line wire was actually just one side of an infinitely large square loop in the left-half-plane. That is to say



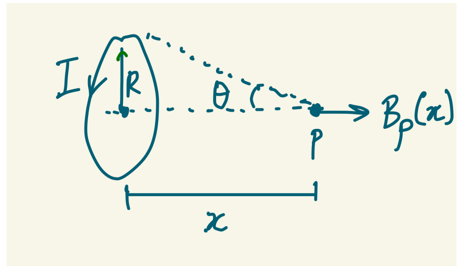

The approx symbol $\approx$ here is in the sense that both currents induce exactly the same magnetic field profile for $x > 0$ (think why). Given this, we now need to find $\phi_{2\to 1}$, the flux linked by the small square loop on the entire left-half-plane.

We can do so by applying the powerful "reciprocity of mutual induction" as follows

$$\boxed{\phi_{2\to 1} = \phi_{1\to 2} = MI = \frac{\mu_0 I a}{2\pi} \ln\left(1 + \frac{a}{d}\right)}$$

Reciprocity of mutual induction is a very general result applicable to any two planar loops in space.

6. *(Magnetic field outside ring current)* A circular ring of radius $R$ is carrying a current $I$. Calculate the magnetic field on the axis of the loop at a distance $x$ from the center.

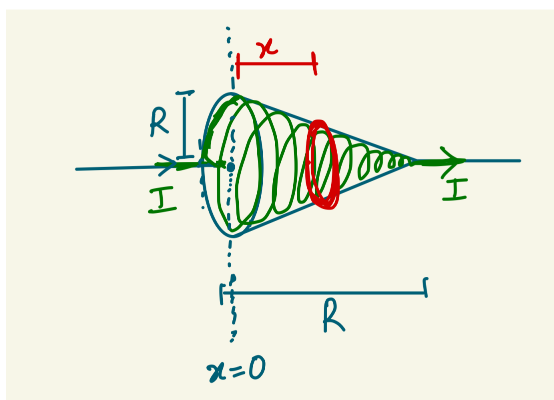



*Solution.* This is a recap of another useful primitive, analogous to a ring charge in electrostatics. Convince yourself that

$$\vec{B}_P(x) = \frac{\mu_0 I R^2}{2} \cdot \frac{1}{(R^2 + x^2)^{3/2}}$$

7. *(Hard self inductance)* A wire is coiled uniformly with $N$ turns on the surface of a cone of radius $R$ and height $R$ too, as shown in the figure. Calculate the self-inductance of the *very first loop in this coil*, which turns around the base of the cone.

[I originally set this problem to find the self-inductance of the whole coil, which turned out exceedingly hard analytically, so a slightly modest sub-problem]




*Solution.* Since base radius and height are same, the radius at a distance $x$ from the centre is simply $R(x) = (R - x)$. This general loop is marked in red color in the figure.

Denote the area of the red loop $A(x)$ and the magnetic field at its center $B(x)$. What is $A(x)$? Easy to see that $A(x) = \pi(R-x)^2$. What is $B(x)$? Using the result from Problem 6, convince yourself that

$$B(x) = \frac{\mu_0 N I}{2R} \int_{u=0}^{u=R} du \; \frac{(R-u)^2}{[(R-u)^2 + (x-u)^2]^{3/2}}$$

The above also uses the uniformity of turns: $dI(x) = I \cdot \frac{N}{R}\, dx$. An involved calculation yields

$$B(x) = \frac{\mu_0 N I}{4\sqrt{2}R} \left[ \ln\left( \frac{x + R + \sqrt{2(x^2+R^2)}}{(\sqrt{2}-1)(R-x)} \right) + \sqrt{2}\left(1 - \frac{R-x}{\sqrt{x^2+R^2}}\right) \right]$$

In particular, we have

$$B(0) = \frac{\mu_0 N I}{4\sqrt{2}R} \ln\left( \frac{\sqrt{2}+1}{\sqrt{2}-1} \right)$$

We know $A(0) = \pi R^2$. Hence the self-inductance of the very first loop is given by

$$\boxed{L(0) = \frac{B(0)A(0)}{I} = \frac{\mu_0 N \pi R}{4\sqrt{2}} \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)} \tag{14}$$

That's it from me. Good luck for your final exams!