Quantum Capacity of channels with small environment

Sriram Gopalakrishnan

Department of Physics
Indian Institute of Technology Madras

April 26, 2019

Introduction

- Michael Wolf et. al. (2007) [1]
- Every *Quantum Channel* T is a completely positive map defined by $\rho \to T(\rho) = \mathrm{Tr}_E[U(\rho \otimes \rho_E)U^\dagger]$.
  An equivalent representation being in terms of Kraus operators.

$$T(\rho) = \sum_{i=1}^{d_E} A_i \rho A_i^\dagger \qquad \sum_i A_i^\dagger A_i = I$$

- We define its *conjugate channel* as $\tilde{T} = \mathrm{Tr}_S[U(\rho \otimes \rho_E)U^\dagger]$.
  The Kraus operators $\tilde{A}_i$ of $\tilde{T}$ are related to those of T by

$$(\tilde{A}_i)_{kl} = (A_k)_{il}$$

Christopher King et. al. (2005) [2]

Quantum Capacity

- *Quantum Capacity* $Q(T)$ is the maximum number of qubits that can be communicated through a quantum channel T, per use of the channel, asymptotically.
- The Quantum Capacity theorem [3] states that

$$Q(T) = \lim_{n\to\infty} \frac{1}{n} \sup_{\rho} J(T^{\otimes n}, \rho) \tag{1}$$

  where a new quantity called *coherent information* is introduced as

$$J(T, \rho) = S(T(\rho)) - S(\tilde{T}(\rho)) \tag{2}$$

- $S(\rho)$ is the standard Von-Neumann Entropy given by

$$S(\rho) = -\operatorname{Tr}(\rho \log_2 \rho) \tag{3}$$

Challenges in evaluating quantum capacity

- Evaluating $Q(T)$ is a challenging task in general since
  - J is not a globally concave function in general.
  - The regularization $n \to \infty$ is necessary since J is not subadditive in general.
- However these obstacles can be avoided for channels with small environment, with the main tool being the *degradability* of the channel.

Degradability of a channel

- A channel T is said to be *degradable* if it can simulate its conjugate. This is in the sense that $\exists$ a channel ϕ such that $\tilde{T} = \phi \circ T$.
- Similarly, a channel T is *anti-degradable* if $\tilde{T}$ is degradable, i.e. $\exists$ a channel Ω s.t. $T = \Omega \circ \tilde{T}$.

Lemma

If T is a degradable channel, then $J(T, \rho)$ is subadditive and concave, and hence $Q(T) = \sup_\rho J(T, \rho)$. If T is anti-degradable, then $Q(T) = 0$.

Dual condition for degradability

- First note that
  Degradability of T $\iff$ complete positivity of $\phi = (\tilde{T} \circ T^{-1})$ and
  Anti-degradability $\iff$ complete positivity of ϕ^{-1}
- Now recall that Channel - State Duality [4] assigns a unique bipartite state $\tau = (T \otimes I)(\omega)$ to each map T, where $\omega = \sum_{i,j=1}^{d} |ii\rangle\langle jj|$ is an unnormalized maximally entangled state. (Choi-Jamiolkowski isomorphism)
- Further, corresponding to each such state τ is a unique *Transfer Matrix* τ^{Γ} defined by

$$\langle ij|\tau^{\Gamma}|kl\rangle = \langle ik|\tau|jl\rangle \tag{4}$$

Dual Condition for degradability

- Two results emerge from the correspondence described in the previous slide:
  - There is a bijective correspondence between maps and transfer matrices.
  - Complete positivity of $T \iff \tau^{\Gamma} \geq 0$
- Degradability of T is equivalent to
  Complete Positivity of $\phi = (\tilde{T} \circ T^{-1}) \iff \tau_{\phi} = [\tilde{\tau}^{\Gamma}(\tau^{\Gamma})^{-1}]^{\Gamma} \geq 0$
- Similarly anti-degradabality of a channel T is equivalent to positivity of $\tau_{\phi^{-1}}$

Qubit Channels

- Now we restrict attention to $d = d_E = 2$, i.e a single qubit system with a single qubit environment.
- According Ruskai et. al. (2002) [5], two channels T and T' have the same capacity if they differ just by unitaries at the input and output.
$$T'(\rho) = VT(U\rho U^\dagger)V^\dagger \tag{5}$$
- Every such channel has a normal form in terms of the Kraus operators
$$A_1 = \begin{pmatrix} \cos\alpha & 0 \\ 0 & \cos\beta \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & \sin\beta \\ \sin\alpha & 0 \end{pmatrix} \tag{6}$$
- Observe that for $\alpha = \beta$, we have a bit-flip channel. And for $\alpha = 0$, we have an amplitude damping channel with $p = \sin^2\beta$

Qubit Channels

- Computing the matrices τ_ϕ and $\tau_{\phi^{-1}}$ and determining their eigenvalues, it is found that their spectra are as follows

$$spec(\tau_\phi) = \{0, 0, \lambda_1, \lambda_2\}, \qquad spec(\tau_{\phi^{-1}}) = \{0, 0, \tilde{\lambda_1}, \tilde{\lambda_2}\} \tag{7}$$

where

$$\frac{\lambda_1}{\lambda_2} = -\frac{\tilde{\lambda_1}}{\tilde{\lambda_2}} = \frac{\cos 2\alpha}{\cos 2\beta} \tag{8}$$

- Now note that $\mathrm{Tr}(\tau_\phi) = \mathrm{Tr}(\tau_{\phi^{-1}}) = d > 0$ since ϕ and ϕ^{-1} are trace preserving. Hence for both matrices, atmost one eigenvalue is negative.
- This leads to conclude

$$\tau_\phi > 0 \iff \frac{\cos 2\alpha}{\cos 2\beta} > 0, \qquad \tau_{\phi^{-1}} > 0 \iff \frac{\cos 2\alpha}{\cos 2\beta} \leq 0 \tag{9}$$

Qubit Channels

- We hence have

$$Q(T) = \begin{cases} \sup_\rho J(T, \rho), & \frac{\cos 2\alpha}{\cos 2\beta} > 0 \\ 0, & \text{otherwise} \end{cases} \tag{10}$$

- Now it is observed that for channels T described by Kraus operators A_1 and A_2,

$$ZT(\rho)Z = T(Z\rho Z) \tag{11}$$

$$Z\tilde{T}(\rho)Z = \tilde{T}(Z\rho Z) \tag{12}$$

Qubit Channels

- To find a supremum for J, note that it is concave in ρ

$$J(\theta\rho_1 + (1-\theta)\rho_2) \geq \theta J(\rho_1) + (1-\theta)J(\rho_2) \tag{13}$$

For $\theta = \frac{1}{2}$, $\rho_1 = \rho$, $\rho_2 = Z\rho Z$, we have

$$J(\rho) \leq J\left(\frac{1}{2}(\rho + Z\rho Z)\right) \tag{14}$$

- Noting that $(\rho + Z\rho Z)$ is always diagonal, WLOG we can substitute

$$\rho = p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1| \tag{15}$$

to maximize J.

The Result

- Substituting this ρ in the equation for $J(T,\rho)$, we hence have

$$Q(T) = \max_{p\in[0,1]}[h(p\cos^2\alpha + (1-p)sin^2\beta) - h(p\sin^2\alpha + (1-p)\sin^2\beta)] \quad (16)$$

wherever $\frac{\cos 2\alpha}{\cos 2\beta} > 0$.

- $h(p)$ is the binary shannon entropy

$$h(p) = -p\log_2 p - (1-p)\log_2(1-p) \quad (17)$$

The Result

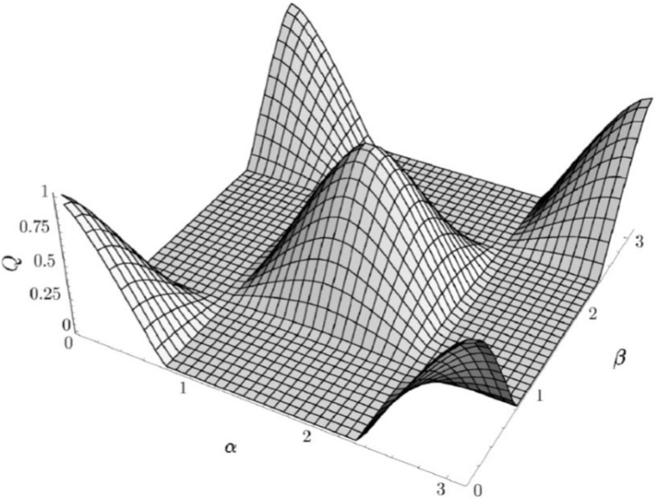

Figure 1: The quantum capacity of extremal qubit channels, as derived by Michael Wolf et. al. [1]

References I

- Michael M Wolf and David Perez-Garcia. "Quantum capacities of channels with small environment". In: *Physical Review A* 75.1 (2007), p. 012303.
- Christopher King et al. "Properties of conjugate channels with applications to additivity and multiplicativity". In: *arXiv preprint quant-ph/0509126* (2005).
- Igor Devetak. "The private classical capacity and quantum capacity of a quantum channel". In: *IEEE Transactions on Information Theory* 51.1 (2005), pp. 44–55.
- Min Jiang, Shunlong Luo, and Shuangshuang Fu. "Channel-state duality". In: *Physical Review A* 87.2 (2013), p. 022310.
- Mary Beth Ruskai, Stanislaw Szarek, and Elisabeth Werner. "An analysis of completely-positive trace-preserving maps on M2". In: *Linear Algebra and its Applications* 347.1-3 (2002), pp. 159–187.

References II

Igor Devetak and Peter W Shor. "The capacity of a quantum channel for simultaneous transmission of classical and quantum information". In: *Communications in Mathematical Physics* 256.2 (2005), pp. 287–303.

Thank You