Summary:
Quantum Capacties of channels with small evironment

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This is a summary of a 2007 paper by Michael Wolf *et. al.* [1]. The paper derives a closed-form expression for the quantum capacity of channels with a two dimensional system and an effectively two dimensional environment. The key technique being used is the degradabality of such channels. Possible extensions of the technique to channels with unconstrained environment are discussed.

Key Points:: Quantum Capacity, Degradability, Single qubit channels.

I. INTRODUCTION

In both Classical and Quantum Information theory, the *capacity* of a channel is an important quantity that provides an upper bound on the number of bits or qubits that can be reliably transmitted in a single use of the channel, and which can be achieved. Classically, the noisy channel coding theorem [2] enables calculation of the channel capacity of all classical channels. However there is no universal coding theorem for quantum channels.

In this summary, the quantum capacity of *extremal qubit channels* are derived. Extremal qubit channels are extreme points of the space of completely positive trace preserving maps (The set of *extreme points* of a convex set are those which cannot be expressed as a convex combination of other points in the set, an hence form a characteristic basis for the set). The authors have found that extremal qubit channels are a particular subset of channels with a two dimensional system and an effectively two dimensional environment. It is important to understand that the effective dimension of a space is not necessarily the same as its physical dimension. For instance, $\text{span}\{|00\rangle, |11\rangle\}$ is effectively two dimensional, but has a physical dimension of 4 (two qubits).

The summary is structured as follows - In section 2, some preliminaries, including the Quantum Capacity theorem is introduced. In section 3, the notion of *degradability* of a quantum channel and its correspondence with a matrix is discussed. In section 4, the idea of degradability is used to derive a closed form expression for the capacity of a 2D-2D channel, and possible extensions to systems with unconstrained environment is discussed.

II. BASICS

Every quantum channel T is a completely positive trace preserving map on the space of $d \times d$ density matrices, described by $\rho \to T(\rho) = \text{Tr}_E[U(\rho \otimes \rho_E)U^\dagger]$. Here ρ_E, the initial state of the environment, is a pure state with dimension $d_E \leq d^2$. An equivalent representation of a quantum channels is in terms of its *Kraus Operators* A_i as -

$$T(\rho) = \sum_{i=1}^{d_E} A_i \rho A_i^\dagger \qquad \sum_i A_i^\dagger A_i = I \tag{1}$$

The condition $\sum_i A_i^\dagger A_i = I$ emerges from the trace preserving nature of the map.

Similar to $T(\rho)$, the *conjugate channel* of a channel is defined by a CPTP map $\tilde{T}(\rho) = \text{Tr}_S[U(\rho \otimes \rho_E)U^\dagger]$. The difference, as can be seen is that the trace is taken over the system basis rather than the environment basis. The Kraus operators $\tilde{A}_i$ of $\tilde{T}$ are related to those of T by $(\tilde{A}_i)_{kl} = (A_k)_{il}$ [3].

Quantum Capacity $Q(T)$ is defined as the maximum number of qubits that can be communicated through a quantum channel T, per use of the channel, asymptotically. The Quantum Capacity theorem, a seminal result formulated by Shor and Devetak [4], states that the quantum capacity has a regularized form in terms of *coherent information* $J(T, \rho)$

$$Q(T) = \lim_{n\to\infty} \frac{1}{n} \sup_\rho J(T^{\otimes n}, \rho) \tag{2}$$

The coherent information of a state ρ with respect to a map T is defined as

$$J(T, \rho) = S(T(\rho)) - S(\tilde{T}(\rho)) \tag{3}$$

where $S(\rho) = -\text{Tr}(\rho \log_2 \rho)$ is the Von-Neumann entropy of ρ. Notice that in the regularized form described by (2), the argument ρ is in a higher dimensional Hilbert space $\mathcal{H}^{\otimes n}$, in which it can potentially be entangled. And this explains why the regularization is necessary. If ρ was a tensor product of identical states, J would become subadditive, thereby eliminating the regularization. The regularization hence gives a kind of space average of the number of qubits that can be transmitted through the channel, by considering what can be transmitted through $T^{\otimes n}$.

Evaluating $Q(T)$ is a challenging task in general because firstly, $J(T, \rho)$ is not always globally concave in ρ, making the evaluation of the supremum a difficult task. Secondly there are channels for which $\sup_\rho J(T^{\otimes n}, \rho)$ is not additive [5]. However the authors show that these obstacles can be surpassed for extremal qubit channels due to their property of *degradability*.

III. DEGRADABILITY

A channel T is said to be *degradable* if it can simulate its conjugate. This is in the sense that $\exists$ a channel ϕ such that $\tilde{T} = \phi \circ T$. Similarly, a channel T is *anti-degradable* if $\tilde{T}$ is degradable, i.e. $\exists$ a channel Ω s.t. $T = \Omega \circ \tilde{T}$.

If a channel is degradable, then J becomes a conditional entropy, which is subadditive and concave. However if a channel is anti-degradable, the fact that $T(\rho)$ cannot be cloned from $\tilde{T}(\rho)$ implies $Q(T) = 0$.

Now an equivalent condition for degradability in terms of a special matrix is developed. First note that degradability of T is equivalent to the complete positivity of $\phi = (\tilde{T} \circ T^{-1})$, and anti-degradability of T is equivalent to complete positivity of ϕ^{-1}. Now recall that Channel - State Duality [6] assigns a unique bipartite state $\tau = (T \otimes I)(\omega)$ to each map T, where $\omega = \sum_{i,j=1}^{d} |ii\rangle\langle jj|$ is an unnormalized maximally entangled state. Further, corresponding to each such state τ is a unique *Transfer Matrix* τ^{Γ} whose matrix elements are defined by

$$\langle ij|\tau^{\Gamma}|kl\rangle = \langle ik|\tau|jl\rangle \tag{4}$$

The result of this mathematics is that there is unique correspondence between channels and transfer matrices. Additionally, the complete positivity of a map T is equivalent to the positive semi-definiteness of its transfer matrix τ^{Γ}. And degradability of a channel T reduces to the positive semi-definiteness of

$$\tau_{\phi} = [\tilde{\tau}^{\Gamma}(\tau^{\Gamma})^{-1}]^{\Gamma} \tag{5}$$

Similarly anti-degradability of T reduces to checking positive semi-definiteness of $\tau_{\phi^{-1}}$.

IV. QUBIT CHANNELS

Equipped with the tools described in the previous section, attention is now restricted to extremal qubit channels, which have dimensions $d = d_E = 2$. The authors show that all extremal qubit channels are degradable or anti-degradable.

According Ruskai et. al. (2002) [7], two channels T and T' have the same capacity if they differ just by unitaries at the input and output.

$$T'(\rho) = VT(U\rho U^{\dagger})V^{\dagger} \tag{6}$$

Using this fact, the authors now claim that equivalence classes of extremal qubit channels with the same capacity have a normal form in terms of Kraus operators with exactly two parameters ($\alpha, \beta \in \mathbb{R}$)

$$A_1 = \begin{pmatrix} \cos\alpha & 0 \\ 0 & \cos\beta \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & \sin\beta \\ \sin\alpha & 0 \end{pmatrix} \tag{7}$$

It is easy to observe that for $\alpha = \beta$, we have a bit-flip channel. And for $\alpha = 0$, we have an amplitude damping channel with $p = \sin^2\beta$.

Since the set of invertible channels is dense (in the topological sense), it suffices to consider invertible channels among those parametrized by (7), and a more general capacity result can be derived by continuity [8].

Computing the matrices τ_{ϕ} and $\tau_{\phi^{-1}}$ and determining their eigenvalues, it is found that their spectra are as follows

$$spec(\tau_{\phi}) = \{0, 0, \lambda_1, \lambda_2\} \tag{8}$$

$$spec(\tau_{\phi^{-1}}) = \{0, 0, \tilde{\lambda_1}, \tilde{\lambda_2}\} \tag{9}$$

where

$$\frac{\lambda_1}{\lambda_2} = -\frac{\tilde{\lambda_1}}{\tilde{\lambda_2}} = \frac{\cos 2\alpha}{\cos 2\beta} \tag{10}$$

Note that $\mathrm{Tr}(\tau_{\phi}) = \mathrm{Tr}(\tau_{\phi^{-1}}) = d > 0$ since ϕ and ϕ^{-1} are trace preserving. Hence for both matrices, atmost one eigenvalue is negative. Together with (10), it leads us to say that either $\tau_{\phi} \geq 0$ or $\tau_{\phi^{-1}} \geq 0$. In other words, extremal qubit channels are always either degradable or anti-degradable. We hence have

$$Q(T) = \begin{cases} \sup_{\rho} J(T, \rho), & \frac{\cos 2\alpha}{\cos 2\beta} > 0 \\ 0, & \text{otherwise} \end{cases} \tag{11}$$

Now it can be verified that for channels T described by Kraus operators A_1 and A_2,

$$ZT(\rho)Z = T(Z\rho Z) \tag{12}$$

$$Z\tilde{T}(\rho)Z = \tilde{T}(Z\rho Z) \tag{13}$$

To find a supremum for J, note that it is concave in ρ whenever $\frac{\cos 2\alpha}{\cos 2\beta} > 0$.

$$J(\theta\rho_1 + (1-\theta)\rho_2) \geq \theta J(\rho_1) + (1-\theta)J(\rho_2) \tag{14}$$

For $\theta = \frac{1}{2}$, $\rho_1 = \rho$, $\rho_2 = Z\rho Z$, we have

$$J(\rho) \leq J\left(\frac{1}{2}(\rho + Z\rho Z)\right) \tag{15}$$

Noting that $(\rho + Z\rho Z)$ is always diagonal, we can substitute

$$\frac{(\rho + Z\rho Z)}{2} = p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1| \tag{16}$$

in the RHS of (15) to maximize J. We hence have

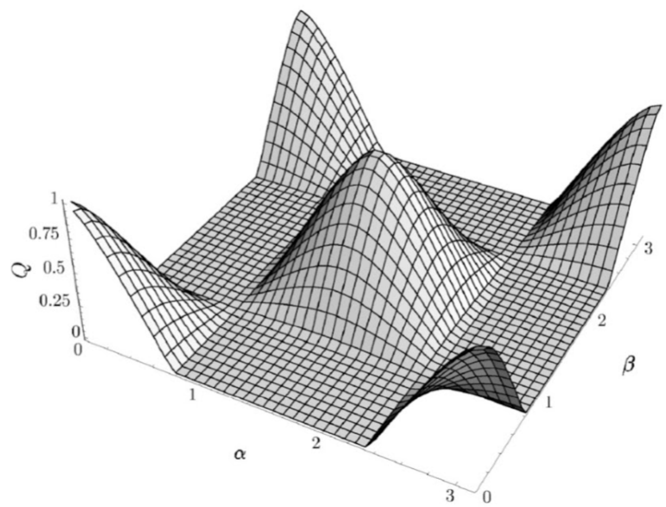

FIG. 1. The quantum capacity of extremal qubit channels. Source: Michael Wolf *et. al.* (2007) [1]

$$Q(T) = \max_{p\in[0,1]} [h(p\cos^2\alpha + (1-p)sin^2\beta) - h(p\sin^2\alpha + (1-p)\sin^2\beta)] \quad (17)$$

for $\frac{\cos 2\alpha}{\cos 2\beta} > 0$, and zero otherwise.

$h(p)$ here is the binary shannon entropy

$$h(p) = -p\log_2 p - (1-p)\log_2(1-p) \quad (18)$$

The capacity is plotted (Fig 1) as a function of α and β for both parameters in the interval $[0,\pi]$. As can be seen from the plot, a maximum capacity of 1 is attained for $(\alpha,\beta) = (0,0),\ (0,\pi),\ (\pi,0),\ (\frac{\pi}{2},\ \frac{\pi}{2}),\ (\pi,\pi)$. And a zero capacity is observed in the rectangular regions $\alpha \in [\frac{\pi}{4}, \frac{3\pi}{4}]$ & $\beta \in [0,\frac{\pi}{4}] \cup [\frac{3\pi}{4},\pi]$ as well as $\beta \in [\frac{\pi}{4}, \frac{3\pi}{4}]$ & $\alpha \in [0,\frac{\pi}{4}] \cup [\frac{3\pi}{4},\pi]$, corresponding to $\frac{\cos 2\alpha}{\cos 2\beta} \leq 0$.

V. REMARKS

Possible extensions of this result to channels with larger environment relies on the potential convexity of Quantum Capacity. Using Eq (2), it can be shown that in the set of degradable channels, the additivity of Quantum Capacity $Q(\otimes_i T_i) = \sum_i Q(T_i)$ implies convexity $Q(pT_1 + (1-p)T_2) \leq pQ(T_1) + (1-p)Q(T_2)$.

Hence arbitrary qubit channels, which can be expressed as a convex combination of extremal qubit channels, would then have simple upper bounds in terms of the bound obtained in Eq (17). However the general behaviour of Quantum Capacity under mixing is not known, and hence additivity of Quantum Capacity is not necessarily a reasonable assumption.

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