Postulates of Quantum Mechanics

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Dirac Notation

1. Column vector or wavefunction: "ket" : eg. $|v\rangle = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ or $e^{-x^2/2}$
2. Row vector: "bra" : $\langle v| =$ transpose conjugate of $|v\rangle$
3. Inner Product: $u^T v \equiv \langle u|v\rangle$
4. Outer Product: $uv^T \equiv |u\rangle \langle v|$
5. "Joint" state/Tensor product: $|u\rangle \otimes |v\rangle \equiv |uv\rangle$

   where $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \otimes \begin{pmatrix} 3 \\ 4 \end{pmatrix} \equiv \begin{pmatrix} 3 \\ 4 \\ 6 \\ 8 \end{pmatrix}$

Operations can be generalized for functions.

The 1st postulate: Hermiticity

- Physical Observables (x, p, H etc) are eigenstates (aka basis states) of Hermitian Operators ($H = H^\dagger$)
- eg. The Schrodinger Equation is an eigenvalue problem for the Hamiltonian operator

$$H |n\rangle = E_n |n\rangle$$

- The span of these basis states is called a Hilbert Space.

The 2nd and 3rd postulates: Superposition and Measurement

- A system can be prepared in a Superposition of operator eigenstates.

$$|\psi\rangle = \sum_n c_n |n\rangle$$

- Normalization implies

$$\langle\psi|\psi\rangle = 1 = \sum_n |c_n|^2$$

- The Measurement of a superposition leads to a random projection onto one of the basis states, say $|k\rangle$ with probability $|c_k|^2$

The 4th postulate: Entanglement

- Two or more isolated systems can be Entangled.
- Say System A $\equiv$ span$\{|0\rangle_a, |1\rangle_a\}$ and System B $\equiv$ span$\{|0\rangle_b, |1\rangle_b\}$.
- "Joint" System AB $\equiv$ span$\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$
- Then the following is an eg of an entangled joint state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

- On the other hand,

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle) = \frac{1}{\sqrt{2}} |0\rangle_a \otimes (|0\rangle_b + |1\rangle_b)$$

is not entangled.

The 5th and 6th postulates: Unitary Evolution and Completeness

- The time evolution of a quantum state is always Unitary.
- For a finite dimensional vector space, this means that one can manipulate a state only using Unitary Matrices ($U^\dagger U = I$)
- Example: The Pauli matrices:

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

- Completeness: The Hilbert Space of any physical operator is "complete".

$$\sum_n |n\rangle \langle n| = I$$