The Tent Map

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Outline

Introduction

What is a tent map ?

1. The tent map is defined by: $T_r(x) = r\min\{x, 1-x\}$ $(0 \le x \le 1)$
2. Is a valid map only when $0 \le r \le 2$.
3. For $r < 1$, stable f.p at origin. For $r = 1$, the entire interval $[0, 0.5]$ is fixed. For $r > 1$, there are unstable fixed points at the origin and $\frac{r}{r+1}$.

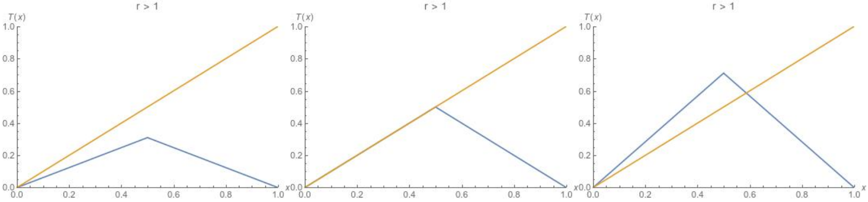


Figure 1: Tent map for various ranges of r in [0,2].

Introduction

Tent map at $r = 2$

1. Iterates of the map $T_2^n(x)$ for $n \geq 2$ have a simple geometric structure.
2. $T_2^n(x)$ has 2^{n-1} tents in [0,1], implying 2^n intersections with $y = x$ line, and hence the map has 2^n period-n points.
3. #Prime period-n orbits = $(2^n -$ (prime periodic points with period = factors of n)$)/n$

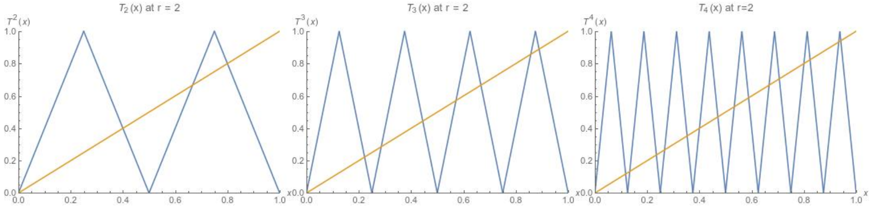


Figure 2: Iterates of the map at $r = 2$

Introduction

Interesting observation

1. 2 prime period-2 points and 4 prime period-4 points are created at $r = 1$.
2. Conjecture: One prime periodic orbit of period 2^k is born at r = 1 for all natural numbers k.

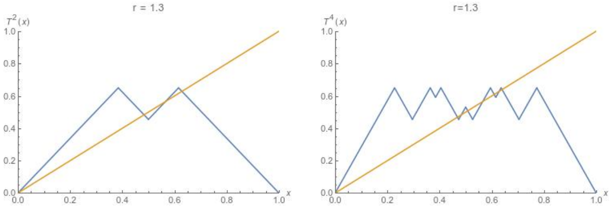


Figure 3: The plots show the creation of period-2 and period-4 points as r is increased slightly beyond 1.

Birth of Period three orbit

Period 3 implies chaos

1. A prime period-3 orbit is born at $r \approx 1.62$, and there are two prime period-3 orbits for $1.62 < r \leq 2$.
2. By Li and Yorke's theorem, the existence of period-3 points implies the existence of periodic points of all periods, as well as the existence of an uncountable subset of $[0, 1]$ that is neither periodic, nor asymptotically periodic.

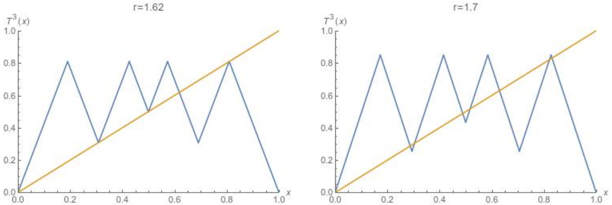


Figure 4: The plots show the creation of prime period-3 points at $r = 1.62$

Definition of Chaos

Devaney's definition of Chaos

Let X be a metric space. A continuous map $f : X \to X$ is said to be *chaotic* on X if -

1. f is topologically transitive, i.e for every pair of non-empty open sets U and V in X, there is a non-negative integer n such that $f^n(U) \cap V \neq \emptyset$.
2. The periodic points of f are dense in X.
3. f has sensitive dependence on initial conditions.

Due to Banks et. al. (1992), it has been established that conditions 1 and 2 imply condition 3.

Bifurcation, Sensitive Dependence

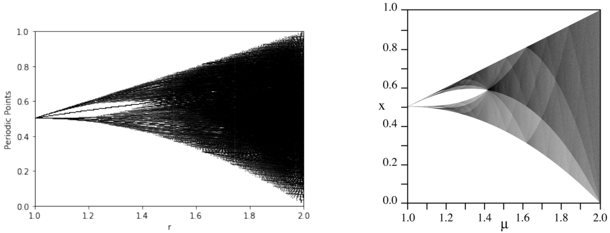


Figure 5: Left: Our approximate bifurcation (Python) Right: Wikipedia

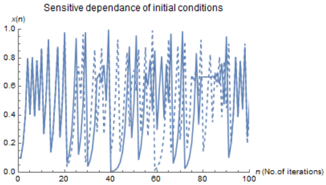


Figure 6: $x_{01} = 0.1$, $x_{02} = 0.1000001$, N = 100

Invariant Density

Frobenious Peron Equation

1. The Frobenius-Peron evolution of phase space density for T_2 is given by : $\rho(x, n+1) = \frac{1}{2}\ \rho\left(\frac{x}{2}, n\right) + \frac{1}{2}\ \rho\left(1 - \frac{x}{2}, n\right)$
2. The functional equation governing invariant densities of the map is hence : $\rho_0(x) = \frac{1}{2}\ \rho_0\left(\frac{x}{2}\right) + \frac{1}{2}\ \rho_0\left(1 - \frac{x}{2}\right)$
   It turns out that the invariant density of T_2 is $\rho_0(x) = 1$.
3. NOTE: The simulation was done for r=1.999.

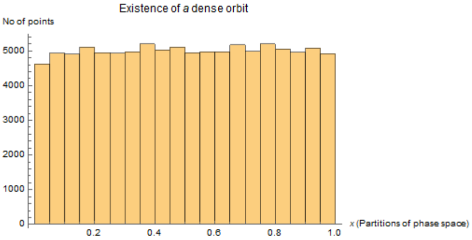


Figure 7: $x_0 = 0.1$,N= 100,000.

Interesting properties of tent map

Properties at r=2

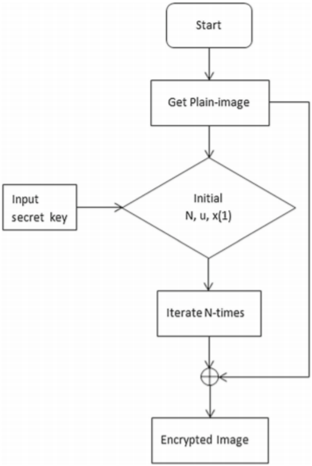
1. A point $c \in [0,1]$ is **periodic iff** it is a **rational number** with **even numerator and odd denominator**, (i.e) $c = \frac{p}{q}$ where p is odd and q is even. eg. $\frac{2}{7}$
2. The other two types of rationals (odd/odd and odd/even) are **asymptotically periodic** (eg. $\frac{3}{8}$), while irrationals are aperiodic.
3. $T_2(x)$ is **topologically conjugate** to $L_4(x)$ via $h(x) = \sin^2(\frac{\pi x}{2})$.(i.e) $T_2 \circ h = h \circ L_4$. As discussed in class, topological conjugacy leads to an interesting insight: Since the condition can be extended as $T_2^n \circ h = h \circ L_4^n \quad \forall n \geq 1$, and because $\sin^2(\frac{\pi x}{2})$ in injective in $[0,1]$, there is a one-to-one correspondence between periodic orbits of T_2 and L_4 for all periods. Here we use the Logistic map as $L_r(x) = r\ x(1-x)$.

An application of tent map

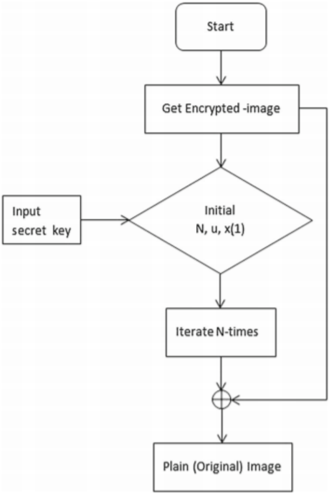
Tent map in Image encryption

1. The chaotic properties of the tent map are used to come up with an algorithm to encrypt an image for upholding the security of the same.
2. This method was proposed in Chunhu Li et al. "An image encryption scheme based on chaotic tent map".

Encryption and Decryption algorithms



(a) Encryption



(b) Decryption

Figure 8: Flowcharts of the algorithms

Experimental results

Testing the algorithm on "Lenna's" image

1. The following experimental results were presented to demonstrate the efficiency of the proposed image encryption algorithm.
2. The standard grayscale image Lenna (Fig. 9a) with the size (256 X 256) pixels was used for the experiment. The results of the encryption are presented in Fig. 9b.
3. In this scheme, we take the key to the original as follows: $x_0 = 0.000001$, $r = 1.999999$.
4. When taking the wrong key, the difference between wrong and right key is 10^{-16}. For example, using $x_0 = 0.0000009999999999$ as the wrong key to decrypt the encryption image, we got a wrong decrypted image shown in Fig. 9f.

Experimental results

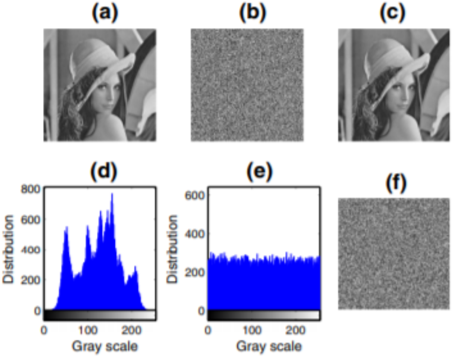


Figure 9: a) The original image. b) The encrypted image. c) The decrypted image. d) The histogram of original image. e) The histogram of ciphered image. f) The decrypted image with wrong key.

References

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THANK YOU